

THE EXPANSION OF SPACE: FREE PARTICLE MOTION AND THE COSMOLOGICAL REDSHIFT

by Alan B. Whiting

Cerro Tololo Inter-American Observatory

The meaning of the expansion of the universe, or the ‘expansion of space,’ is explored using two phenomena: the motion of a test particle against a homogeneous background and the cosmological redshift. Contrary to some expectations, a particle removed from the Hubble flow never returns to it. The cosmological redshift is not an ‘expansion effect;’ in special cases it can be separated into a kinematic (special relativistic) part and a static (gravitational redshift) part, but in general must be thought of as the effect on light rays of their propagation through curved space-time. Space as such does not affect matter by ‘expanding,’ but only through its curvature.

Introduction: Comparing two cosmologies

The acceptance of General Relativity (GR) in place of the work of Newton marked a great change in the concepts and language of gravitational physics. Space itself was taken from its previous role as the unobtrusive stage on which everything happened, to become a major player in the game: it was measured and characterised by the metric tensor, and it could curve and vibrate. In the particular case of cosmology, space defined by a network of comoving coordinates became an expanding rubber sheet, or perhaps a river, carrying matter along with it in its motion.

At the same time Newtonian physics is still an extremely accurate approximation of GR as long as local speeds are small and gravitational fields (space curvature) are weak. In these regions it should give similar, if not identical, results to a full relativistic treatment.

But the idea of expanding space is hard to fit into the Newtonian picture. Do galaxies in the Hubble flow move apart because they have a certain initial velocity and an inertia, or because they are carried along in expanding space? This is not an idle question. If one thinks of a particle which is *not* part of the background cosmology, the answer one gets for its qualitative motion will depend on how one thinks of space. The first part of the following study is a mathematical investigation of this problem: using free-particle motion in an attempt to show just what is happening as the universe expands.

The second part deals with the cosmological redshift. It is normally portrayed as a direct consequence of cosmological expansion, and thus might be taken to show immediately the reality of the ‘expansion of space.’ However, kinematical derivations of it also appear, resulting in the same formulae. We can expect, then, that clarifying the physical source (assuming we can give that phrase a meaning) of the cosmological redshift will also tell us something about the nature of the expansion of space.

Background Cosmology

First we must specify the background universe mathematically. In both Newtonian and relativistic cases we begin with a homogeneous, isotropic distribution of pressureless matter (‘dust’); and, with the exception of massless test particles and occasional light rays propagating through, we will keep it that way.

1. Newtonian forms

We begin with the Newtonian derivation*. Since all points in the universe are equivalent, we pick one arbitrarily as the centre and origin. Around it there is a sphere of radius a , filled with matter of density ρ (which will change with time, but not location). If the universe around the sphere is isotropic, all external mass will have no gravitational effect, and we may ignore it. We choose a such that the mass M internal to the sphere remains the same always. The dynamic equation for a is

$$\begin{aligned}\ddot{a} &= -\frac{4\pi G}{3}\rho a \\ &= -\frac{GM}{a^2}\end{aligned}\tag{1}$$

One change (only) can be made to this equation which still allows a homogenous, isotropic universe to remain so: a term linear in a can be added to the right-hand side. Within Newtonian physics there is no reason to include it. But to allow later comparison, we will insert such a ‘cosmological constant’ Λ and occasionally use it (to conform to the relativistic convention, it appears here as $\Lambda/3$):

$$\begin{aligned}\ddot{a} &= -\frac{4\pi G}{3}\rho a + \frac{\Lambda}{3}a \\ &= -\frac{GM}{a^2} + \frac{\Lambda}{3}a\end{aligned}\tag{2}$$

This constant has the same effect in Newtonian cosmology as in relativistic versions: it acts to counter the effect of gravity, possibly forming a static (if unstable) universe.

For a full examination of the possible solutions to equation (2) see Bondi (1961)⁴. For our purposes it is sufficient to look at the four which are mathematically most tractable: the overdense, critical and underdense situations with no constant; and the spatially flat case (zero-energy) with the constant.

For the critical case,

$$r = \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}$$

(the mathematical simplicity of this solution has made it more attractive than its physical applicability might warrant; but all solutions approach this one at early times). Analytical solutions for the other three cases are given in the appendix.

2. Relativistic forms

*There is no *a priori* reason to expect that classical dynamics should be able to calculate an entire universe, much less come up with the right answers for its motion. But the idea was investigated by Milne (1934)¹ and extended by McCrea (1955)² in response to criticism by Layzer (1954)³; and Bondi (1961)⁴, especially pp. 104-5, affirms its validity.

In the language of General Relativity the assumptions of isotropy and homogeneity require a space described by the Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dx^2}{1 - x^2/R^2} + x^2 d\theta^2 + x^2 \sin^2 \theta d\phi^2 \right) \quad (3)$$

where x , θ , ϕ are spherical spatial coordinates, t is the time coordinate and R is the (perhaps imaginary) radius of curvature of three-dimensional slices of the manifold orthogonal to time. An alternative formulation uses the dimensionless constant $k = 1/R^2$; k positive, negative and zero corresponds to positive, negative and zero three-dimensional curvature. If x is given units of length, a is unitless (and will be treated so here). The speed of light c is taken to be unity.

We must now figure out what these coordinates mean. In GR (as in special relativity) there is a basic ambiguity in the words ‘distance’ and ‘time,’ in that durations and separations can be quite different when measured by different observers. There is no absolute Newtonian framework to which one may refer. In the general case one *cannot* define the distance between two objects at a given time, because no two observers will agree on a common time.

Our universe, however, is not the general case. Homogeneity and isotropy make it extremely symmetric. Because of its symmetries it is possible to construct a cosmic time function, t , on which all observers stationary with respect to the spatial coordinates in Equation (3) will agree[†].

For a given value of t , then, we may construct a three-dimensional hypersurface, and on this hypersurface find the proper distance between two points by integrating ds in the above formula. Without loss of generality we may orient the coordinate system so our points lie on the same value of the angles, giving the expression

$$ds = \frac{adx}{\sqrt{1 - x^2/R^2}} \quad (4)$$

Alternatively, this expression can be simplified by using the substitution

$$y = \begin{cases} R \arcsin(x/R) & R^2 > 0 \\ \rho \operatorname{arcsinh}(x/\rho) & R^2 < 0, R = i\rho \end{cases}$$

to give the simpler formula $ds = ady$. For flat cosmologies, $x = y$. Thus these coordinates provide an unambiguous measure of (infinitesimal) time and distance. Homogeneity guarantees that all our observers will agree on them[‡].

Now we specify a matter field *motionless* with respect to the coordinates x, θ, ϕ , having a mass-energy density ρ . (This is important to notice: we have defined a coordinate system tied to our objects.)

From the field equations we then obtain

$$\begin{aligned} 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2 R^2} - \Lambda &= 0 \\ \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2 R^2} - \frac{\Lambda}{3} &= \frac{8\pi G\rho}{3} \end{aligned} \quad (5)$$

[†]For a more complete discussion of cosmic time, symmetries, and spacelike hypersurfaces, see Misner, Thorne & Wheeler 1973⁵, sections 27.3 and 27.4

[‡]Observers moving with respect to this coordinate system will, of course, have different time and distance measurements.

Combining these we may get an equation for \ddot{a} identical with that derived in the Newtonian case; the arbitrary constant which is identified with Newtonian energy is here related to spatial curvature. The solutions are the same, though the interpretation of some quantities differs.

In order to compare relativistic with Newtonian results, we need to make a connection between the coordinate systems. Thanks to the symmetries of our cosmological system, identifying the Newtonian absolute time t with the cosmic time function t presents no problems. But there are several possible ways to extend the Newtonian distance r into curved space. We want to choose the most natural extension, that is, the one which preserves most of the characteristics of the simpler system. We would therefore like a relativistic distance which can be defined entirely on a spatial hypersurface and one which gives the same physical results (that is, having the same relations with other physical quantities, at least on a small scale). Unfortunately, we cannot be completely satisfied.

Very often in cosmology one uses the comoving distance, Δy from the formulae above. It is convenient in that objects which are stationary with respect to the general matter-field (most objects in our universe) have fixed comoving coordinates; thus one may easily keep track of them, and the expansion of the universe (which is sometimes a nuisance, at least mathematically) is factored out. The comoving distance is also defined on a spatial hypersurface. But a fixed comoving volume contains a varying density of particles and radiation, an effect which could be detected locally in (for example) a varying temperature. One would be led to abandon, perhaps, the law of conservation of mass-energy, leading to a rather complicated sort of physics. More generally, comoving coordinates map to Newtonian distances only in the limit of stationary a ; they are a zero-order, not a first-order match. For purposes of dynamics, they are not a useful way to generalise the concepts of distance and motion[§].

Another alternative is the radar distance, the time taken by a light signal to pass between two observers (converted into distance using c). This matches well with a familiar way to measure distance. But it depends on a process of some extension in time, and thus cannot be defined on a spacelike hypersurface; and the volumes described by it do not quite correspond to those which conserve mass-energy.

The measure to be used here is the proper distance ay in the formulae above. It is restricted to a hypersurface (and thus one can easily find its temporal derivative, and so describe motion); and the volumes it describes allow conservation of mass-energy. Unfortunately, it does not correspond to the radar distance: two objects of constant proper distance, in an expanding universe, will have a decreasing radar distance. We shall accept this, and proceed.

With this identification, note that the two theories have come up with precisely the same results. Compared to any observation, they will give the same prediction. In fact it seems rather metaphysical to argue whether (on the one hand) two points are actually moving apart, or (on the other) the space between them itself is growing[¶].

To make some headway we will have to drop a test mass into the middle and see what happens. If the ‘expansion of space’ acts like a normal Newtonian force, we would expect the test mass to join in the Hubble

[§]Odenwald & Fienberg (1993)⁶, using comoving coordinates, decided that the cosmological redshift can have no source in relative motion; meanwhile there is a derivation (which will appear below) which traces such redshift *directly* to relative motion. One is at liberty to define one’s quantities as convenient, but the results may lack applicability.

[¶]Compare the following Zen story: Two monks were arguing about a flag. One said, ‘The flag is moving.’ The other said, ‘The wind is moving.’ The sixth patriarch happened to be passing by. He told them, ‘Not the wind, not the flag; mind is moving.’ From Reps & Senzaki (1955)⁷.

flow more or less immediately; something like this is what Harrison (1995)⁸, for example, assumes (without explicit calculation). At the very least, a mass at rest or close to it should start to move in the direction of the flow.

If the expansion of space acts more like a viscous force, the test particle should move with respect to the Hubble flow at an ever-diminishing speed, and will (perhaps asymptotically) reach a situation in which it and the background coordinate system at its position are moving with the same velocity.

The explicit calculation of what actually happens is our next task.

Free Particle Motion

Our interest is in a particle of negligible mass placed within some background cosmology, but not constrained to move along with the dust which determines it^{||}. For this the equations of motion will require a slight modification. Using r as the radial coordinate of our free particle^{**}, the Newtonian form of the dynamical equations looks like

$$\begin{aligned}\ddot{r} &= -\frac{GM}{r^2} + \frac{\Lambda}{3}r \\ &= -\frac{4\pi G}{3}\rho r + \frac{\Lambda}{3}r\end{aligned}\tag{6}$$

(the Λ term is included or excluded, depending on the background cosmology). This time, however, the mass M within r is not constant but depends on the motion of the test particle. The density ρ is a function of time, but not of r , and depends on the cosmology; we will thus work with the second of equations (6). There will be four general solutions, one corresponding to each of our four backgrounds.

It is easier to visualise what is going on by specifying a certain set of boundary conditions. In this case we will specify that the free particle is placed at some distance r_0 from the origin and held there motionless, that is, kept at a constant value of ay ; then at time t_0 it is released. This is not as restrictive a circumstance as it might appear. If a particle is given any velocity which takes it out of the flow, there is some place still in the flow with the same velocity, and which we may take as an origin. Any subsequent motion with respect to the flow will take place along the line between them, which we may choose as our radial coordinate. Plotting radial distance versus time thus gives a very general look at any motion of a free particle.

For a critical universe, the density goes like

$$\rho = \frac{1}{6\pi G t^2}$$

which gives the general solution as

$$r = c_1 t^{2/3} + c_2 t^{1/3}$$

^{||}Free particle motion, using Newtonian calculations in one type of background universe, is treated in Peacock (2001)⁹. Davis, Lineweaver & Webb (2003)¹⁰ investigate the same problem; however, they make certain errors along the way which lead to unfortunate conclusions, difficult to interpret in a physical way. These will be discussed below. The problems with their work, in fact, led to my efforts.

^{**}Again, based on the FRW coordinate system. One tied to the particle itself would, of course, look rather different.

and again setting c_2 to zero gives comoving motion. A free particle at rest at r_0 at time t_0 moves as

$$r = 2r_0 \left(\frac{t}{t_0} \right)^{1/3} - r_0 \left(\frac{t}{t_0} \right)^{2/3}$$

and is plotted in Fig. (1). The overdense solution is plotted in Fig. (2) and the underdense case in Fig. (3). Analytic expressions for the latter two may be found in the Appendix.

In each case without the cosmological constant, the free particle accelerates toward the origin *away* from the Hubble flow, passes through the origin, and continues out the other side. In the unbound cases it recedes indefinitely, and can be shown to cross the zero coordinate exactly once in all its travels. In the closed-universe case it turns around after the universe itself begins to collapse, and passes through the origin at exactly the moment of singularity (as it should, since otherwise it will have found a way to escape its universe entirely). Here also it can be shown that the free particle has a zero coordinate exactly once per cycle of the background universe (apart from singular times).

If the universe has a cosmological constant, the initial behaviour of the free particle depends on the balance between gravitational attraction and cosmic repulsion. If the constant wins, the free particle accelerates away from the origin; otherwise, it moves in.

This behaviour of a free particle is not what one would expect, if the ‘expansion of space’ acts like a Newtonian force pushing the galaxies apart. It is *qualitatively* different, the initial velocity being in the opposite direction.

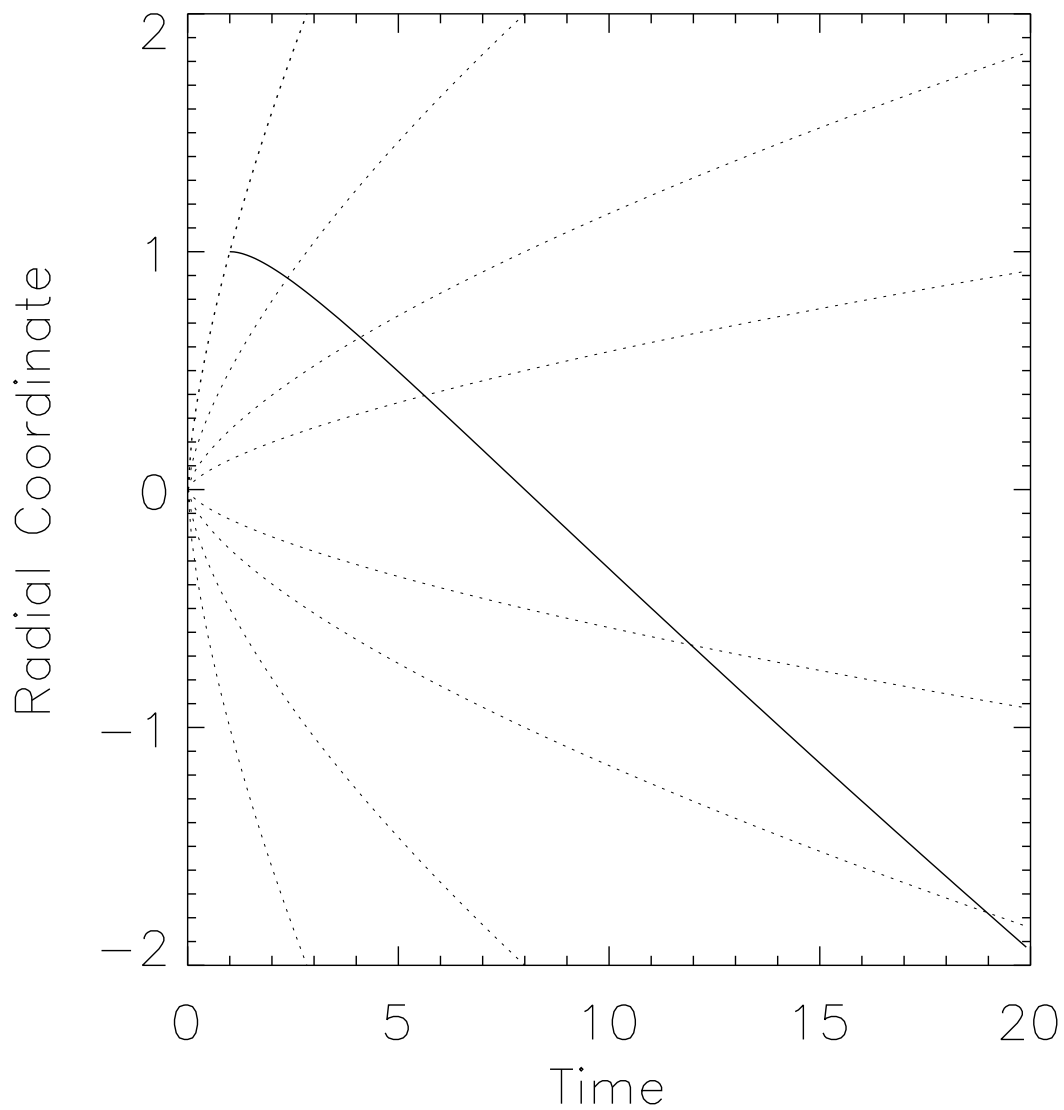


Fig. 1.— Motion of a free particle (solid line) in a critical-density universe without cosmological constant. Dashed lines show the trajectories of some comoving particles. Radial coordinates are shown as a function of time, with the free particle at rest at some initial time t_0 . As before, dashed lines show the trajectories of some comoving particles. The free particle accelerates toward the origin, passes through it and continues out indefinitely.

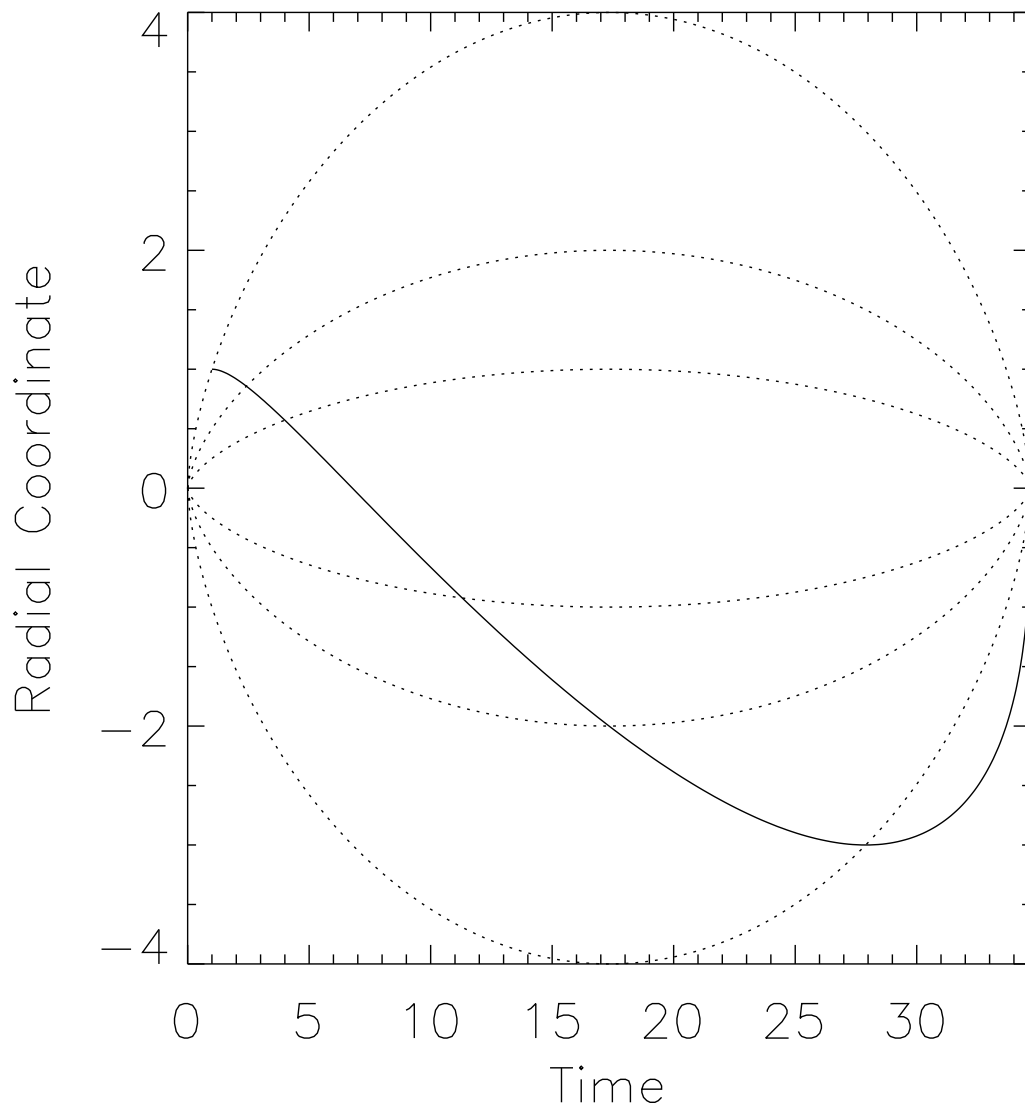


Fig. 2.— Motion of a free particle (solid line) in an overdense universe without cosmological constant. As before, dashed lines show the trajectories of some comoving particles. The free particle is at rest at time t_0 , which is during the expansion phase; then accelerates toward the origin, passes through it and continues out, turning around after the universe starts to collapse. It reaches the origin again at the moment of singularity.

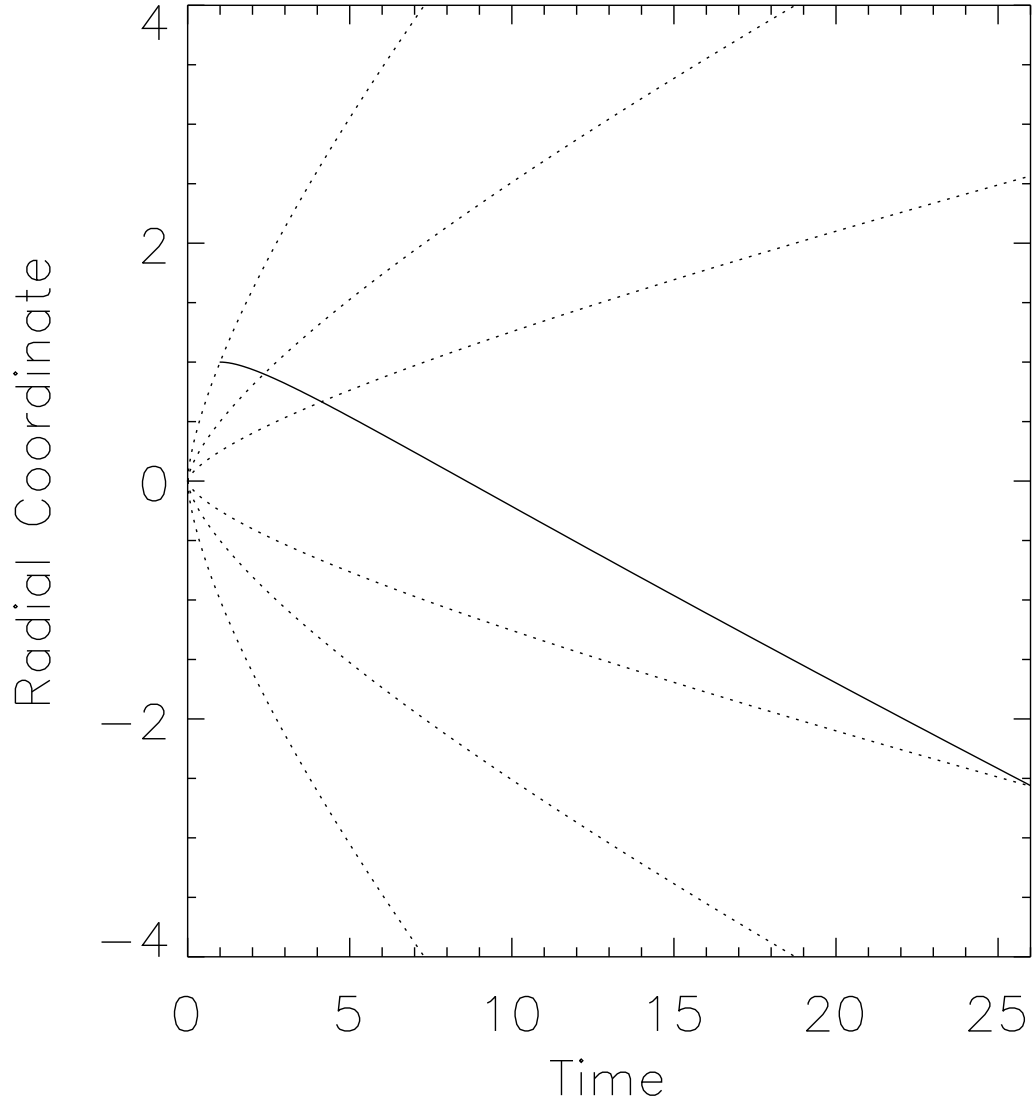


Fig. 3.— Motion of a free particle (solid line) in an underdense universe without cosmological constant. As before, the free particle accelerates toward the origin, passes through it and continues out indefinitely.

There are two objections to accepting the above arguments as disproof of the idea of flowing space. One is that the derivations here are entirely Newtonian, and so of course will not show an effect which is relativistic in origin; that problem will be addressed below. The second is that an origin placed a finite distance away from the test particle is not appropriate, and one should really consider the problem to be that of a particle given an initial peculiar velocity. In that case, the Hubble flow could be thought of as exerting a sort of viscous force. One would then not necessarily expect the expansion of space to act immediately and overwhelmingly. While a leaf placed in a river might become part of the flow at once, a ship has inertia, and once the engines are stopped will continue to move with respect to the water for some time (as anyone who has attempted fine maneuvering knows very well). And it does appear, since peculiar velocity is known to decay with time, that the free particle indeed joins the Hubble flow after passing through the origin. It is to the matter of joining the Hubble flow that we now turn.

The decay of peculiar velocity

It is a general property of expanding universes that any motion which departs from the general flow decreases with time, the familiar decay of peculiar velocity^{††}. In the solutions above (and in the Appendix) it is readily seen that the time derivatives of the c_2 functions are all monotonically decreasing (with the exception of the overdense universe in some phases), and indeed decreasing more quickly than the c_1 (Hubble flow) functions. From this it seems clear that all particles must eventually join the Hubble flow as their velocity away from it vanishes. However, this is worth investigating quantitatively.

Considering the motion of a free particle in a critical universe without a cosmological constant, its asymptotic speed is

$$v_\infty = -\frac{2}{3}r_0 \frac{t^{-1/3}}{t_0^{2/3}}$$

The background particle with this same asymptotic speed has as its equation of motion

$$a = -\left(\frac{t}{t_0}\right)^{2/3}$$

The spatial distance between these two particles is thus

$$r - a = 2r_0 \left(\frac{t}{t_0}\right)^{1/3}$$

which is unbounded. For an underdense universe, the distance between the free particle and the background particle with the same asymptotic speed (we might call it a ‘peculiar distance’) is more complicated, but as time becomes large approaches a constant. That is, the free particle stays at least this far from its corresponding background particle, even at infinite times. A particle which *always* keeps a distance from its ‘proper’ place in the Hubble flow, even after an infinitely long time, cannot really be said to join it.

For the overdense situation there is of course no asymptotic solution. The free particle never approaches any particular value of speed or comoving coordinate, so cannot be said to join the Hubble flow.

^{††}Lightman et al. (1975)¹¹, problems 19.19 and 19.20, deal with the asymptotic fate of free particles in underdense and critical universes, and in fact have more elegant derivations than the ones I am about to produce. However, mine may make the phenomena of the decay of peculiar velocities a bit clearer, especially in the context of the question I am trying to answer.

In the case of a flat universe with a cosmological constant, a particle removed from the flow in the limit of late times *does* have an asymptotically decreasing peculiar distance.

Thus it cannot be asserted that it is a general characteristic of cosmological models for free particles to be swept into the Hubble flow, even asymptotically. Peculiar velocities do vanish eventually in expanding universes; but so do *all* velocities. As time goes on everything winds down, like a motion picture shown by a projector attached to an old battery. This does not mean that the villain will wind up attached to the leading lady.

It is true that in the flat, cosmological-constant situation a nonrelativistic particle disturbed from the Hubble flow in the limit of late times returns to it. This special case can hardly be taken to show anything general about the nature of spacetime[†].

Relativistic free-particle motion

To determine the motion of a free particle in General Relativity, one solves the geodesic equation:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \quad (7)$$

which relates the second derivatives of the coordinates x^i with respect to proper time s along the path to the curvature (via the connection coefficients Γ) and the first derivatives of the coordinates. By invoking symmetry and setting up our coordinates so that the test particle moves only along the x^1 axis (the radial spatial coordinate), the geodesic equation for a particle moving in the general metric (equation (3)) is

$$\begin{aligned} \frac{d^2 x}{ds^2} + \frac{x/R^2}{1 - x^2/R^2} \left(\frac{dx}{ds} \right)^2 + \\ 2 \left(\frac{\dot{a}}{a} + \frac{x\dot{x}/aR^2}{1 - x^2/R^2} \right) \frac{dx}{ds} \frac{dt}{ds} = 0 \end{aligned}$$

Unfortunately, s is not a particularly useful variable for comparing with Newtonian results or with observation. We may convert to time as the independent variable by using equation (3) and dividing through by ds^2 ; at the same time, we will simplify things by going to the previously defined comoving radial variable y . This results in

$$\ddot{y} = a\dot{a}\dot{y}^3 - 2\frac{\dot{a}}{a}\dot{y} \quad (8)$$

where superposed dots denote derivatives with respect to coordinate time.

The comoving radial variable y is still not directly comparable with anything in the Newtonian derivation. We therefore convert to the proper distance $r = ay$, which transforms equation (8) into

$$\ddot{r} = \frac{\dot{a}}{a} \left(\dot{r} - \frac{\dot{a}}{a} r \right)^3 + \frac{\ddot{a}}{a} r \quad (9)$$

For small values of the first term on the right-hand side, equation (9) is identical with the corresponding equations derived by Newtonian methods (by way of equations (5)). The solutions are, therefore, identical,

[†]Davis et al.¹⁰ apparently assume, from an inspection of their diagrams equivalent to Figs. (1) and (3), that free particles *do* eventually rejoin the Hubble flow on the far side of the ‘center.’ This shows one danger with relying on numerical calculations (which can never actually go out to infinity) and the associated computer-generated plots.

and the behavior of free test particles is the same as already set out as long as the coordinate speeds are much smaller than that of light. This does not mean we have merely recovered the Newtonian limit of a relativistic situation, and so have approximated away anything interesting. If a particle is to rejoin the Hubble flow by some property of space-time, it should do so at any speed; and in particular should do so when it is close to the flow.

Now consider even relativistic speeds. If a particle's speed is, say, greater than the Hubble flow at its position, the parenthesis of equation (9) is positive; in an expanding universe, \dot{a}/a is positive; so the relativistic correction to its acceleration is also positive, that is, it tends to get even further from the flow. The conclusion of the previous section is reinforced: there is no sign of a flow of space, carrying objects with it; and bodies disturbed from their places in the Hubble flow do not return to it. In fact, if the relativistic terms are included, even the one case in which a particle *did* eventually join the flow is found to be inexact.

From their expressions for free particle motion, Davis et al.¹⁰ conclude that it is not the motion of the background universe which causes the decay of peculiar velocity and thus (in their view) the eventual rejoining of the Hubble flow, but the *acceleration* of that flow. This is a difficult thing to understand physically. How can the acceleration of a coordinate system have a physical effect? But recall that this coordinate system is tied to a set of masses; so perhaps it is from the acceleration of the background mass of the universe that the effect proceeds. There might be an analogy with electromagnetism (in the elementary, nonrelativistic formulation), in which the velocity of a charge causes a magnetic field which may exert a force; and an accelerating charge radiates, thus potentially affecting other objects. But we are not dealing with electromagnetism here. In General Relativity, some motions can cause gravitational radiation and thus move other bodies; but a homogeneous, isotropic situation, such as we have here, produces none. Have Davis et al.¹⁰ discovered a previously unsuspected effect in cosmology, or perhaps physics?

Not in any general sense. Suppose we take $a = kt$ in Equation (8), that is, an expanding but non-accelerating universe. Then the solution for \dot{y} is

$$\dot{y} = \frac{1}{kt\sqrt{1 - c_1 t^2}}$$

with c_1 an arbitrary constant. This peculiar velocity should be identically zero if its source is the acceleration of the Hubble flow; clearly it is not.

Davis et al.¹⁰ did not look at situations like this, confining themselves to universes with more conventional dynamics, all of which have accelerating scale factors. The appearance of \ddot{a} in their expression for free-particle motion suggested to them a causal relationship. In fact, as far as physics goes, the acceleration of a is caused by a matter density (plus, perhaps, a cosmological constant), the same thing which causes free particle acceleration; \ddot{a} is an effect, not a cause[‡].

The details of our chosen distance do not affect this conclusion. It is straightforward to show that, if the radar distance had been chosen instead of the proper spacelike distance ay , the failure of a free particle to match the Hubble flow (in position and velocity) even at infinite times remains.

The Cosmological redshift

[‡]It can be argued that a uniformly expanding universe is unphysical, in that it does not correspond to observation. Surely, however, the *nature* of space cannot depend on the equation of state of the matter placed in it.

Having apparently got rid of the idea of the flow of space, we come up against it in full force when we turn to the cosmological redshift. Authors seem to be in agreement that this phenomenon is a *direct* result of the expansion of the universe. Misner, Thorne & Wheeler⁵, p. 776 and Peebles (1993)¹², p. 96-7, use the picture of a standing wave with expanding boundary conditions and Rindler (1977)¹³, p. 213 calls it ‘an *expansion* effect rather than a *velocity* effect’ (his emphasis). Leaving aside the question of a ‘velocity’ effect for the moment, there are reasons to be uneasy at the picture presented. It is not clear, for instance, *why* there should be a standing wave generated between comoving points in the universe, nor why it should maintain itself. More importantly, as pointed out by Cooperstock et al. (1988)¹⁴ (among others), electromagnetic radiation automatically tracking the universal expansion cannot be right. All our test equipment and comparisons are built of or use electromagnetic forces, and they should also expand with the universe; so any cosmological redshift would be undetectable in principle. At the very least, atoms in the Hubble flow would change their characteristic wavelengths with time (and perhaps with the state of the local gravitational field), leading to strange results indeed.

In spite of the agreement among authors that the cosmological redshift is not a matter of Doppler effects between objects in motion with respect to each other, its mathematical behaviour can be derived on that basis. Following Peebles¹², pp. 94-5, we look at the propagation of a light ray between neighboring comoving observers. If they are close enough, the space between them is close to being flat, and we may use the special relativistic formula for the Doppler shift due to their mutual motion:

$$\frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1 + v/c}{1 - v/c}} \quad (10)$$

(we assume pure radial motions). Since the relative speed v is due to the Hubble flow, for observers a distance δr apart it becomes $H\delta r$; and the first-order change in wavelength is given by

$$1 + \frac{\delta\lambda}{\lambda} = 1 + H\delta r$$

now replacing H by its definition in terms of a ,

$$\frac{\delta\lambda}{\lambda} = \frac{\dot{a}}{a}\delta r$$

and noting that we are treating a light ray, with $r = ct$, we find

$$\frac{\delta\lambda}{\lambda} = \frac{\delta a}{a}$$

that is, the wavelength of the propagating ray is proportional to the scale factor, the standard cosmological result. So it appears that the cosmological redshift *is* a velocity effect, at least when observed locally.

May we then ignore GR entirely? We can do so only in an empty, flat universe. Let us construct one of these, populated with a set of massless observers expanding according to some Hubble law (which may vary with time) $v = Hr$. In this Minkowsky space the redshift seen by two neighboring observers looking at a propagating light ray is of course given by equation (10). However, this time the relative speed v is *not* given by $H\delta r$. The expansion can only be uniform with respect to one point, which for convenience we may call the centre. A particle moving at a speed Hr_1 with respect to the centre, when measured with respect to another particle moving at Hr_2 with respect to the centre, will be seen to be going at the speed

$$v_{12} = \frac{H\delta r}{1 - H^2 r_1 r_2} \quad (11)$$

The relativistic correction depends not on the relative positions of the particles, but on their absolute positions with respect to the centre, and cannot always be made small by taking particles close together. The formula for the cosmological redshift does not work in this universe[§]. It cannot, then, be *wholly* due to velocity.

The difference between the Minkowsky universe and the previous one lies in the presence of some four-dimensional curvature. That allows the neighboring comoving particles all to be equivalent; or alternatively, it keeps all their time axes parallel, in spite of relative motion. Mass is important. That suggests another calculation.

Suppose we try to calculate the cosmological redshift by combining the flat-space Doppler shift with the static gravitational redshift. Assume that we may hold the (isotropic, homogeneous, pressureless) universe still for an instant, so we need only calculate the gravitational redshift in it. We compare two observers, one at the centre of an imaginary sphere of radius r and the other on the surface (just as in the derivation of Newtonian cosmological equations). The gravitational redshift between them is given by

$$\frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1 - GM_o/r_o}{1 - GM_e/r_e}}$$

where the subscript o denotes the observer (on the surface of the sphere) and e the emitter (at the centre; though the light need not be emitted there, only pass through and its wavelength measured). But for a universe of uniform density ρ both r_e and M_e vanish, in such a way as to leave unity on the bottom of the fraction under the radical. M_o can be conveniently computed from the density and radius.

Now ignore gravity for the moment, and consider only the velocity Doppler shift between the centre and a radius r . This is

$$\frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1 + (\dot{a}/a)r}{1 - (\dot{a}/a)r}}$$

Combining the formula for the gravitational redshift with the velocity Doppler shift, we get

$$\frac{\lambda_o}{\lambda_e} = \sqrt{\frac{1 + (\dot{a}/a)r}{1 - (\dot{a}/a)r}} \sqrt{1 - \frac{8\pi G}{3}\rho r^2} \quad (12)$$

Now, in the case of a spatially flat FRW universe with no cosmological constant (only), the dynamic equations (5) allow us to replace $8\pi G\rho/3$ with $(\dot{a}/a)^2$. This gives us

$$1 + \frac{\Delta\lambda}{\lambda} = 1 + \frac{\dot{a}}{a}r$$

and we recover the formula for the cosmological redshift. In this one case the cosmological redshift can be broken down into a factor of velocity alone and one of static gravitation alone. We may in this case attribute its cause to a simple combination of both static mass and motion[¶].

[§]This is not quite the Milne universe, which has many other interesting features; describing them, however, would take us too far afield from the subject of this paper. A good summary may be found in Rindler¹³. Note that the flat, empty universe here is *not* an FRW universe. As may be shown from the dynamic equations (5), an empty FRW universe must either have curvature or a cosmological constant.

[¶]Bondi (1947)¹⁵ found an analogous situation for the rather more complicated situation of the Tolman-Bondi metric. In his formula for overall redshift he could identify effects of motion and something like a static gravitational redshift. However, there were other terms, not so simply interpretable.

There are several reasons to be uneasy over this derivation. For one, a static pressureless universe is impossible; there must either be pressure (enough to show up in the energy-momentum tensor, or possibly as a cosmological constant) or motion. More worryingly, we have both assumed a spatially flat universe (in substituting the dynamics equations in Equation (12), and one that is curved (in using the equation for gravitational redshift); thus the r variables in each part of the derivation are not necessarily the same (though at large distances they certainly approach each other). The result should be taken as no more than indicative that both motion and gravity contribute to the cosmological redshift. Moreover, other situations are not even this simple; so we still seek a more general way of thinking of the phenomenon. For this, we consider a different derivation.

Note that a light ray follows a null geodesic, so that all along it

$$ds^2 = 0 = -dt^2 + \frac{a^2 dx^2}{1 - x^2/R^2} \quad (13)$$

Everywhere along this ray, then,

$$\frac{dt}{a} = \frac{dx^2}{\sqrt{1 - x^2/R^2}} \quad (14)$$

If we integrate this equation along the path of propagation from the emission to the observation of the light ray we get

$$\int_{t_{obs}}^{t_{emit}} \frac{dt}{a} = \int_{x_{emit}}^0 \frac{dx}{\sqrt{1 - x^2/R^2}} \quad (15)$$

which is just the comoving distance between the emitter and the observer.

Now we integrate the expression again, this time starting at a time just one wavelength later (δt_e) and ending at a time just one wavelength later (δt_o). The difference between this integral and the previous one will be the change in comoving distance. But if the emitter and observer are both stationary with respect to the comoving coordinates, there is no change in comoving distance. So

$$\int_{t_{obs}}^{t_{emit}} \frac{dt}{a} = \int_{t_{obs} + \delta t_o}^{t_{emit} + \delta t_e} \frac{dt}{a} \quad (16)$$

And if any change in a is small during the period of one light wave,

$$\frac{\delta t_o}{a_o} = \frac{\delta t_e}{a_e} \quad (17)$$

which is the desired result.

In this derivation we have made use of the fact that light propagates along null geodesics, in a space whose curvature enters implicitly through the function a . In effect, we have constructed the quadrilateral bounded by two null rays (on either side) and two wavelengths (at either end). In a four-dimensionally flat (Minkowsky) universe both ends will have the same length and there is no cosmological redshift. In a universe with any curvature, the null rays will diverge (or focus). Since the scale factor a measures the proper distance between geodesics followed by comoving observers, it also measures the curvature of the universe in a particular way, a way convenient for computing the net change in proper distance between the ends of the quadrilateral.

This is the physical source of the cosmological redshift: the propagation of null rays through curved spacetime. Expansion of a set of comoving observers, or of ‘space’ itself (whatever that might mean), is another effect of curvature, not the cause.

Conclusions

The concept of ‘space’ certainly changed drastically between classical Newtonian dynamics and General Relativity. However similar (sometimes identical) the effects of gravity calculated by each method, the underlying ways of thinking are quite different. The fact that space in GR has an active role in dynamics, however, does not mean it has the attributes of a physical object. It does not act like a viscous fluid, drawing all bodies into the Hubble flow, even asymptotically; it does not affect things by ‘expanding,’ nor by accelerating this expansion. In fact, if one looks at space itself apart from the associated fluid of cosmological matter, it is not at all certain what ‘the expansion of space’ means.

The effect on space on matter is through its curvature. In the mantra of Misner, Thorne & Wheeler⁵, matter tells space how to curve; space tells matter how to move. A free particle in a cosmological background moves according to the curvature of space; the curvature is produced by the background matter. Motion of the background matter (as long as it remains nonrelativistic, contributing nothing to the stress-energy tensor) has no direct effect on curvature; looked at this way, it would be strange indeed if a free particle were somehow constrained to follow that motion. The motion of the background, however, is *also* determined by the curvature of space. This means that the background geodesics measure the curvature (in a way which is particularly convenient for anyone trying to calculate a redshift).

Acknowledgements

The author would like to thank the referees for clarifying several points in this exposition (clarity being of particular importance in this matter). Especially, the matter of distances (radar, peculiar, proper and otherwise) require special care, and their comments helped greatly.

Appendix: analytical solutions to equations of motion

1. The scale-factor

Solving equation (2) in the overdense case, the evolution of a with time is best given parametrically, with both a and time t functions of the ‘development angle’ η :

$$\begin{aligned} t &= \frac{GM}{\sqrt{2}E^{3/2}} \left(\eta - \frac{1}{2} \sin(2\eta) \right) \\ a &= \frac{GM}{E} \sin^2 \eta \end{aligned} \tag{18}$$

where E is the negative of the total specific energy of the test-point on the surface of the sphere.

The underdense solution also is best expressed parametrically in terms of a development angle η :

$$\begin{aligned} t &= \frac{GM}{\sqrt{2}E^{3/2}} \left(\frac{1}{2} \sinh(2\eta) - \eta \right) \\ a &= \frac{GM}{E} \sinh^2 \eta \end{aligned} \tag{19}$$

where this time E is the (positive) total specific energy of the test point at radius a .

The situation with a cosmological constant Λ which is easily handled mathematically occurs when the total energy is zero, in which case

$$a = \left(\frac{6GM}{\Lambda} \right)^{1/3} \sinh^{2/3} \left(\frac{\sqrt{3\Lambda}}{2} t \right)$$

Otherwise, elliptical integrals must be used, and the overall behaviour is not so easily seen.

2. Free-particle motion

For the overdense case, the density varies with time according to

$$\rho = \frac{3E^3}{4\pi G^3 M^2} \frac{1}{\sin^6 \eta}$$

which gives, as the solution for equation(6),

$$r = c_1 \sin^2 \eta + c_2 \sin \eta \cos \eta$$

with c_1 and c_2 undetermined constants. If c_2 is set to zero the free particle is never removed from the Hubble flow. If the particle is released, at rest, a distance r_0 from the origin at a time corresponding to the development angle η_0 , the solution is

$$r = r_0 \frac{\cos 2\eta_0 (1 - \cos 2\eta) - \sin 2\eta_0 \sin 2\eta}{\cos 2\eta_0 - 1}$$

and is plotted in Fig.(2).

For an underdense universe, the density is described by

$$\rho = \frac{3E^3}{4\pi G^3 M^2} \frac{1}{\sinh^6 \eta}$$

and the general solution of equation(6) is

$$r = c_1 \sinh^2 \eta + c_2 \sinh \eta \cosh \eta$$

with c_2 determining peculiar motion as before; the particular solution is

$$r = r_0 \frac{\cosh 2\eta_0 (\cosh 2\eta - 1) - \sinh 2\eta_0 \sinh 2\eta}{\cosh 2\eta_0 - 1}$$

and is shown in Fig. (3).

For the zero-energy/flat universe with antigravity, the density is

$$\rho = \frac{\Lambda}{8\pi G} \frac{1}{\sinh^2 \left(\sqrt{3\Lambda} t/2 \right)}$$

and the general solution to equation (6) is

$$\begin{aligned} r = & c_1 \sinh^{2/3} \left(\sqrt{3\Lambda} t/2 \right) \\ & + c_2 \sinh^{2/3} \left(\sqrt{3\Lambda} t/2 \right) \int \frac{dt}{\sinh^{4/3} \left(\sqrt{3\Lambda} t/2 \right)} \end{aligned}$$

The integral does not appear to be expressable in closed form, which makes working with it somewhat inconvenient. However, some characteristics of it can be derived which are sufficient for the present purpose.

3. The peculiar distance

For an underdense universe, the distance between the free particle and the background particle with the same asymptotic speed is more complicated than in the critical case, but as η becomes large approaches

$$(r - a)_\infty = r_0 \left(\frac{\cosh 2\eta_0 - \sinh 2\eta_0}{\cosh 2\eta_0 - 1} \right)$$

a constant. That is, the free particle stays at least this far from its corresponding background particle, even at infinite times.

In the situation of a flat universe with the cosmological constant, the peculiar motion (c_2) function can be approximated at early times by

$$- \left(\frac{36}{\Lambda} \right)^{1/3} t^{1/3} \quad (20)$$

(plus a constant term from the integral, which can be absorbed into the c_1 function) and at late times by

$$- \frac{3^{1/2}}{2^{2/3} \Lambda^{1/2}} e^{-\sqrt{\Lambda/3} t/2} \quad (21)$$

(with a similar constant consigned to the c_1 function). A particle removed from the Hubble flow at early times thus behaves something like one in a critical universe (which should not be surprising, since *all* FRW universes have the same behaviour at early times), always getting farther from the comoving particle with the same asymptotic velocity. A nonrelativistic particle removed from the flow at late times *does* have a ‘peculiar distance’ which vanishes asymptotically. In this limit of this one situation, then, a body displaced from the universal expansion tends to rejoin it at infinite times.

References

- (1) Milne E. A., Quarterly Journal of Mathematics, Oxford series, V. **5**, 64, 1934
- (2) McCrea W. H., *AJ*, **60**, 271, 1955
- (3) Layzer D., *AJ*, **59**, 268, 1954
- (4) Bondi H., *Cosmology, 2nd edn* (Cambridge University Press, Cambridge), 1961
- (5) Misner C. W., Thorne K. S., Wheeler J. A., *Gravitation* (Freeman, San Francisco), 1973
- (6) Odenwald, S. & Fienberg, R. T. *Sky & Telescope*, **85**, 31, 1993
- (7) Reps, P. & Senzaki, N. *Zen Flesh, Zen Bones* (Doubleday, New York; no date given, but shortly after 1955 based on internal evidence)
- (8) Harrison E. J., *ApJ*, **446**, 63, 1995
- (9) Peacock, J. A., in *Como 2000 Proceedings of the School on Relativistic Gravitation* (Springer, New York), 2001
- (10) Davis, T. M., Lineweaver, C. H., Webb, J. K. *American Journal of Physics*, **71**, 358 (astro-ph/0104349), 2003

- (11) Lightman A. P., Press W. H., Price R. H., Teukolsky S. A., *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton), 1975
- (12) Peebles, P. J. E. *Principles of Physical Cosmology* (Princeton University Press, Princeton), 1993
- (13) Rindler, W. *Essential Relativity (revised second edition)* (Springer-Verlag, New York), 1977
- (14) Cooperstock, F. I., Faraoni, V., Vollick, D. N. *ApJ*, **503**, 61, 1998
- (15) Bondi H., 1947, *MNRAS*, *107*, 411, 1947